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The variational method in contact problems. The present state of the problem and trends in its development☆

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ABSTRACT

Research on variational methods for solving problems on the contact of solid deformable bodies is reviewed and trends in the development of these researches at the present time are analysed. The Signorini problem and it generalizations, numerical methods, different models of friction, investigations into the problem of the existence and uniqueness of a solution, the problem of rolling motion, the problem of describing the boundary conditions, inelastic materials and problems of contact dynamics and electro-elastic contact are considered. The analysis shows that research on the problem of the contact of deformable bodies is being conducted over a broad front in different areas and the results are being applied in different areas of modern engineering and technology.

Several formulations of the problem of the contact between solid deformable bodies exist. By contact problems, in the narrow sense of this word in this paper, we mean problems in which there are constraints in the form of inequalities generated by the dependence of the contact area on external actions and (when account is taken of friction) the existence of a previously unknown interface between the zones of cohesion and slippage. As a rule, problems with a variable contact area are non-linear. Furthermore, even in geometrically linear problems, it is necessary to take account of the change in the shape of the boundaries of the bodies in contact and, here, the solution depends on which way this change in shape is taken into account. Taking account of friction correctly within the limits of even the simplest Amonton–Coulomb law requires the introduction of the relative velocities of slippage and, consequently, calculations of time derivatives and, in quasistatic problems, derivatives with respect to a loading parameter. This effect implies a dependence of the solution on the loading history. Consequently, it is necessary to use step-by-step procedures in the numerical solution. The difficulties lie in the fact that the dependence of the friction forces on the relative slippage rates in the iterative processes in the numerical solution leads to slow convergence of the sequences of friction forces in view of the smallness of the slippage rates.

Results of the solution of the above problems are considered below. A brief historical review is given and several promising areas for development are indicated.

1. Contact problems with zero friction forces: Signorini's problem and its generalizations

1.1. Signorini's problem

The variational method[1] has turned out to be very effective for the mathematical modelling and investigation of contact problems. Lagrange, having constructed a theory of the equilibrium and motion of finite-dimensional systems subjected to bilateral holonomic constraints, that is, to constraints in the form of equations, is the creator of the variational method for solving problems in mechanics. The basis of the method lies in a transition to a relation containing the true and virtual displacements. The mechanical interpretation of this principle involves the fact that the work of forces acting on the system and on any virtual displacements (that is, displacements permitted by the kinematic constraints) is equal to zero. An example is the equilibrium of a small ball in a gravitational force field in a vessel with a smooth bottom. However, when the bottom of the vessel is a cone, displacements along the walls of the cone, for which the work of the forces applied to the system will be positive or, in any case, non-negative, will be permissible in the equilibrium state. This idea was developed by Fourier who showed that the variational Lagrange equation will not be the equilibrium condition for a system with

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unilateral constraints but the corresponding inequality will. We point out that, in both of the examples considered, the equilibrium state will correspond to a stationary point of the total energy of the system which, in the case of stable equilibrium states, will be a minimum point.

Ostrogradskii described a method for solving problems in the dynamics of finite-dimensional systems with unilateral constraints which consists of changing to sequences of problems with bilateral constraints, that is, to the solution of equations. In order to construct these equations, the reactions of the constraints are analysed and, out of the whole set of unilateral constraints, those for which the reaction of the constraints is non-zero are taken into account. Ostrogradskii's method was added to and generalized by Meyer and Zermelo. A detailed account of the history of this problem with a corresponding bibliography is available[2].

In passing from finite-dimensional systems to problems of the mechanics of deformable media, including contact problems, the sums determining the work on virtual displacements are replaced by integrals over the volume occupied by the deformable medium.

The contact problems considered relate precisely to problems for systems with unilateral constraints. This idea was used for the first time by Signorini.[3,4] Signorini's problem involves finding the equilibrium of a linearly elastic body in an absolutely rigid shell without friction. In the initial unstressed and undeformed state, the shell tightly touches a deformable body Ω with a boundary Σ, so that the unilateral constraint, which reflects the requirement that there is no penetration of the points of Ω into the shell, has the form

$$\mathbf{u}(\mathbf{x})\cdot\mathbf{n}|_{\Sigma}\le 0 \tag{1.1}$$

where $\mathbf{u}(\mathbf{x})$ is the displacement vector of a point $\mathbf{x}$ of the body Ω, and $\mathbf{n}$ is the outward unit normal to the surface Σ.

As in classical problems of mathematical physics, we distinguish between the local and variational formulation of contact problems. The local formulation contains:

the equilibrium equations

$$\nabla\cdot\hat{\sigma}+\rho\mathbf{F} = 0 \tag{1.2}$$

where $\hat{\sigma}$ is the stress tensor and the small hat denotes it is of the second rank, $\triangledown$ is the Hamiltonian operator, a dot denotes a scalar product, ρ is the density of the material and F is the density of the mass forces;

the relation between the stresses and strains

$$\hat{\sigma}(t) = \hat{F}\{\hat{\varepsilon}(\tau)\}_{\tau\in[0,t]} \tag{1.3}$$

where t is a parameter determining the change in the state of the system, $\hat{\varepsilon}$ is the tensor of small Cauchy deformations, which is connected to the displacement field $\mathbf{u}$ by the well known relations $\hat{\varepsilon}=(\nabla u+\nabla u^T)/2$, and the curly brackets in formula (1.3) denote that the relation between the stresses and the strains can take account of the history of the change in the deformation process.

Below, as in Signorini's paper, Hooke's law

$$\hat{\sigma} = {}^4\hat{a}\cdot\cdot\,\hat{\varepsilon} \tag{1.4}$$

is used, where ${}^4\hat{a}$ is an elastic moduli tensor.

It is necessary to add the condition for there to be no friction and for of the normal pressure to be non-positive to the above relation. To formulate these conditions, we will represent the force vector $\boldsymbol{\sigma}=\hat{\sigma}\cdot\mathbf{n}$ and the displacement vector $\mathbf{u}(\mathbf{x})$ on the boundary Σ in the form of the sum of the normal (labelled with a subscript N) and tangential (with a subscript T) components

$$\boldsymbol{\sigma} = \hat{\sigma}\cdot\mathbf{n} = \sigma_N\mathbf{n}+\boldsymbol{\sigma}_T, \quad \mathbf{u} = u_N\mathbf{n}+\mathbf{u}_T \tag{1.5}$$

Then, the boundary condition on Σ, reflecting the hypothesis that there are no tensile stresses, will be as follows:

$$\sigma_N(\mathbf{x},t)\le 0, \quad \forall\mathbf{x}\in\Sigma \tag{1.6}$$

The condition for there to be no friction has the form

$$\boldsymbol{\sigma}_T(\mathbf{x},t) = 0, \quad \forall\mathbf{x}\in\Sigma \tag{1.7}$$

Any point of the surface Σ_C is either stress free or in contact with the rigid shell and, consequently,

$$u_N(\mathbf{x})\sigma_N(\mathbf{x},t) = 0, \quad \forall\mathbf{x}\in\Sigma \tag{1.8}$$

Equation (1.8) is called the complementarity condition.

The local formulation of Signorini's problem involves finding the displacements, strains and stresses which satisfy Eqs (1.2) and (1.4) as well as constraints (1.1), (1.6)–(1.8).

The first stage in investigating contact problems is the proof that the local formulation of Signorini's problem is equivalent to the variational inequality

$$a(\mathbf{u},\delta\mathbf{u})\ge L(\delta\mathbf{u}), \quad \forall\delta\mathbf{u} = \boldsymbol{\upsilon}-\mathbf{u}, \quad \mathbf{u}\in K, \quad \boldsymbol{\upsilon}\in K \tag{1.9}$$

where

$$a(\mathbf{u},\delta\mathbf{u}) = \int_\Omega\hat{\sigma}(\mathbf{u})\cdot\cdot\,\hat{\varepsilon}(\delta\mathbf{u})d\Omega, \quad L(\delta\mathbf{u}) = \int_\Omega\rho\mathbf{F}\cdot\delta\mathbf{u}d\Omega \tag{1.10}$$

$$K = \{\boldsymbol{\upsilon}|\boldsymbol{\upsilon}\in V;\ \boldsymbol{\upsilon}\cdot\mathbf{n}\equiv\upsilon_N(x)\le 0,\ \forall\mathbf{x}\in\Sigma\} \tag{1.11}$$

Here, K is the set of kinematically permissible displacement fields and V is the functional space of the solutions. In the given case, V is the space of Sobolev functions possessing generalized first order derivatives. Theorems on the existence and uniqueness of the solution have been successfully proved in precisely this space. Any solution of the local problem satisfies variational inequality (1.9). The inverse assertion only holds with an additional assumption concerning the double differentiability of the solution in a classical sense, which is untrue in the general case. It should be noted that the pioneering work of Signorini was of purely theoretical significance. A new class of problems was discovered for which theorems on existence and regularity were proved and generalizations to multidimensional spaces, etc. were obtained. Some of these results are reflected in a monograph[5] and Signorini's theory was practically completed: theorems on the equivalence of the local and variational formulations and on the transition to the problem of the minimizing the total energy functional were proved within the limits of the apparatus of functional Sobolev spaces and the spaces of traces of functions on a boundary which has been developed up to the present time. Generalizations to some new physical processes such as seepage, heat conduction and control were also given and some problems, taking friction into account, were also solved.

Examples of analytical solutions of contact problems with a variable contact area are given in monographs.[6,7]

1.2. Numerical methods for solving Signorini's problem

Progress in solving specific problems depended on the development of numerical methods. The results of the solution of applied problems, obtained in the period from the end of the 1960's to the end of the 1970's, have been generalized in a monograph.[8] It was proved using the methods of the variational calculus (or the concept of a functional derivative, which is more customary at the present time) that inequality (1.9) is a necessary and sufficient condition for a minimum of the total energy functional

$$J(\mathbf{v}) = \frac{1}{2}a(\mathbf{v},\mathbf{v}) - L(\mathbf{v}) \tag{1.12}$$

in the set of functions K.

The requirement of the strict convexity of the bilinear form $a(\upsilon,\upsilon)$ is, from the point of view of mechanics, the most important constraint in the proof of the assertion which had been formulated:

$$a(\mathbf{v},\mathbf{v}) \ge \alpha\|\mathbf{v}\|_v^2, \quad \alpha = \text{const} > 0 \tag{1.13}$$

where V is the space of the solutions, $\|\upsilon\|$ is a norm in V and, moreover, the functional $\mathrm{L}(\upsilon)$ must be continuous in V.

In view of the importance of this problem in areas such as design optimization and optimal control, an enormous number of papers have been devoted to solving the problem of minimizing functional (1.12) with constraints. The majority of these methods are based on reducing the initial problem to a sequence of minimization problems without constraints which are solved, for example, by the method of steepest descents.

Thus, the following iterative process is used:

a zeroth approximation of the solution $\mathbf{u}=\mathbf{u}^{(0)}\in K$ is given;

the direction of steepest decrease (descent) of the functional being minimized from the point $\mathbf{u}^{(0)}$ is found; as is well known, this direction is identical to the antigradient of the functional (1.12), that is, the element $\mathrm{grad}J(\mathbf{u}^{(0)})$ for which

$$J'(\mathbf{u}^{(0)},\boldsymbol{\phi}) = \langle \mathrm{grad}J(\mathbf{u}^{(0)}),\boldsymbol{\phi}\rangle, \quad \forall\boldsymbol{\phi}\in V \tag{1.14}$$

is determined, $J'(\mathbf{u},\boldsymbol{\phi})$ denotes a Gâteaux derivative of the functional $J(\mathbf{u})$ at the point $\mathbf{u}$ along the direction of φ and the angular brackets denote a scalar product in the space V. In practice, finding the gradient reduces to solving a problem in the theory of elasticity without unilateral constraints;

the first approximation is calculated using the formula

$$\mathbf{u}^{(1)} = \mathbf{u}^{(0)} - \beta\,\mathrm{grad}J(\mathbf{u}^{(0)}) \tag{1.15}$$

where β is a parameter controlling the rate of convergence;

it is checked that the element $\mathbf{u}^{(1)}$ belongs to the set K. If $\mathbf{u}^{(1)}$ does not belong to K, then one returns to the permissible set K. This last operation is achieved using different methods: most often (in the case of fixed β) an orthogonal projection of the element onto K is used.

The convergence of this process is ensured by inequality (1.13) and the convexity of the set K. This property also holds exactly in Signorini's problem. Here,

$$u_N^{(1)} = \begin{cases} 0, & \text{if} \quad u_N^{(1)} > 0 \\ u_N^{(1)} = u_N^{(1)}, & \text{if} \quad u_N^{(1)} \le 0 \end{cases}$$

A theoretical justification of this algorithm has been given[8] and the solutions of two-dimensional contact problems have been obtained.[9]

Other methods for solving problems involving minimization of functionals of the type (1.12) have been described.[8]

1.3. Generalization of Signorini's problem

The first significant generalization of Signorini's problem, associated with the introduction of an initial gap $\boldsymbol{\delta}_N$, was proposed[5] without a detailed treatment of the question regarding the calculation of the quantity $\boldsymbol{\delta}_N$. This problem was initially solved[10] for the case when one of the contacting bodies is absolutely rigid and, later, for the arbitrary case, with an estimate of the accuracy of the different forms of non-penetration. It was proved that a contact problem is equivalent to the problem of minimizing a functional of the form (1.12), where now

$$K = \{\mathbf{v}\,|\,\mathbf{v}\in V;\ \mathbf{v}(\mathbf{x}) = 0,\ \forall\mathbf{x}\in\Sigma_u;\ \upsilon_N(\mathbf{x}) \le \delta_N(\mathbf{x})\ \forall\mathbf{x}\in\Sigma_C\} \tag{1.16}$$

where $\boldsymbol{\Sigma}_C$ is the area of possible contact, $\boldsymbol{\delta}_N(\mathbf{x})$ is the distance from a point $\mathbf{x}$ on the surface $\boldsymbol{\Sigma}_C$ to the corresponding point on the surface of a rigid punch along the normal to $\boldsymbol{\Sigma}_C$. Here, one of the possible forms of the gap measure, which is convenient for applications is used and $\boldsymbol{\Sigma}_u$ is the part of the boundary $\boldsymbol{\Sigma}$ of the domain Ω in which the displacements are equal to zero.

The linear form $L(\upsilon)$ now has the form

$$L(\boldsymbol{\upsilon}) = \int_{\Omega} \rho \mathbf{F} \cdot \boldsymbol{\upsilon} d\Omega + \int_{\Sigma_\sigma} \mathbf{P} \cdot \boldsymbol{\upsilon} d\Sigma \tag{1.17}$$

where $\boldsymbol{\Sigma}_{\boldsymbol{\sigma}}$ is the part of the boundary $\boldsymbol{\Sigma}$ of the domain Ω at which surface forces with a density P are specified.

Progress in developing methods for solving contact problems has been made by using variational transformations - the Young–Fenchel–Moro method. A complete set of variational principles is obtained including analogues of the principles of Castigliano, Reissner, etc.[11] The version in which the problem of minimizing the functional (1.12) (with the substitution of (1.17)) on the set (1.16) is transformed into the problem of finding the saddle point

$$J(\upsilon) + \int_{\Sigma_c} \sigma_N(\delta_N - \upsilon_N) d\Sigma \to \sup_{\sigma_N \le 0} \inf_{\boldsymbol{\upsilon} \in V} \tag{1.18}$$

has been found to be the most effective version from this set for solving contact problems.

The generalized Udzawa algorithm[8] is used to solve this problem. In this algorithm, initial distributions of the contact interaction forces are first specified, after which the displacements are calculated by solving a regular problem (without one-sided constraints) in the theory of elasticity and then checking the non-penetration condition. In the case when the non-penetration condition is not satisfied, the contact interaction forces are corrected by means of a shift along the gradient with respect to the contact interaction forces. The requirement that the contact pressure should not be positive is satisfied by a projection of the permissible contact interaction forces onto the set. The combination of these two steps leads to the following formula for the next approximation

$$\sigma_N^{(1)} = P_N(\sigma_N^{(0)} + \rho_{0N}(\delta_N - u_N^{(0)})); \quad P_N(\sigma_N) = \begin{cases} \sigma_N, & \sigma_N \le 0 \\ 0, & \sigma_N > 0 \end{cases}$$

in which $\boldsymbol{\rho}_{0N}$ is a numerical parameter, that controls the convergence. A detailed description of the algorithm is available.[11]

Many two-dimensional and three-dimensional contact problems, which are important for applications, have been solved using this method (Refs 11–13, etc.).

2. Taking account of friction forces

The problem of taking friction forces into account when the contact area and the boundary of the slippage and cohesion zones depends on external actions does not, as a rule, lend itself on analytical solution in view, above all, of its non-linearity. Moreover, the friction forces in the zones where sliding occurs depend on the relative rates of slippage of the contacting bodies. In many cases, these rates are very small while it is precisely these rates that are used in iterative processes based on satisfying (in the limit) velocity-type laws of friction, but it is only such laws which have a physical meaning.

In order to construct the solutions, additional hypotheses are introduced which allow one to surmount the above-mentioned difficulties and which naturally have a limited range of applicability. These hypotheses basically reduce to the following (hypotheses which are different from those enumerated below and sometimes contradict one another are used in different papers):

the contact area is known,

the boundary of the zones of slippage and cohesion is specified,

the normal pressure is independent of the friction forces,

the law of friction only contains displacements and does not contain velocities.

Many theoretical and applied problems have been solved using these hypotheses. A review of them is available.[14–16] Note that the extremely restricted hypotheses which have been formulated do not, as a rule, enable analytical solutions to be found. The solution obtained by Galin,[17] which is reconstructed in Spence's papers,[18] is one of the rare examples of analytical solutions.

One of the most important technical problems in which the solutions of contact problems with friction are used is the prediction of fracture (the formation and propagation of cracks) in the case of fretting, that is, the cyclic loading of structural components in contact. The solutions and corresponding techniques required for such a prediction have been gathered together in a monograph.[19]

It was found that only the variational approach and the methods for implementing it enabled an almost complete solution of a contact problem to be obtained which is free from the restrictive hypotheses listed above.

A method, which enables the problem to solved using velocity-type laws of friction was proposed for the first time by the author[20] and several modifications of the method have been described.[11,20] A step-by-step procedure using the simplest (Eulerian) type of loading parameter is employed. It is proved that, in the case of the Amonton–Coulomb law of friction, the power of the friction forces is estimated from above in the following manner

$$W_f \le \int_{\Sigma_C} f|\sigma_N(\mathbf{u})||\dot{\mathbf{u}}_T| d\Sigma \tag{2.1}$$

where f is the coefficient of friction and a dot denotes a derivative with respect to the loading parameter.

Another idea, proposed in Ref. 5 for another formulation of a contact problem with friction involves the use of the limiting equality

$$f|\sigma_N||\boldsymbol{\upsilon}_\mathbf{T}| = \max_{\mu_\mathbf{T},\, |\mu_\mathbf{T}| \le \mathbf{f}|\sigma_\mathbf{N}|} (\boldsymbol{\mu}_\mathbf{T} \cdot \boldsymbol{\upsilon}_\mathbf{T}) \tag{2.2}$$

Estimate (2.1) and formula (2.2) enabled it to be proved that the local formulation of the problem of the contact of an absolutely rigid fixed punch with a deformable body Ω when friction is taken into account is equivalent to the inequality

$$a(\mathbf{u}, \delta\dot{\mathbf{u}}) + \int_{\Sigma_c} f|\sigma_N(\mathbf{u})|(|\dot{\boldsymbol{\upsilon}}_T| - |\dot{\mathbf{u}}_T|)d\Sigma \geq L(\delta\dot{\mathbf{u}}), \quad \forall\delta\dot{\mathbf{u}} = \dot{\boldsymbol{\upsilon}} - \dot{\mathbf{u}}, \quad \dot{\boldsymbol{\upsilon}} \in \dot{K}_u \tag{2.3}$$

where

$$\dot{K}_u = \{\dot{\boldsymbol{\upsilon}} | \dot{\boldsymbol{\upsilon}} = \dot{\mathbf{u}} + \delta\dot{\mathbf{u}};\ \Psi(\alpha)'_\alpha \cdot (A^{-1} \cdot \delta\dot{\mathbf{u}}) \geq 0,\ \forall\mathbf{x} \in \Sigma_C^t\} \tag{2.4}$$

$$\Sigma_C^t = \left\{\mathbf{x} | \mathbf{x} \in \Sigma_C;\ \Psi(\alpha(\mathbf{x})) = 0;\ \hat{A}' \cdot (\dot{\hat{A}}^{-1} \cdot (\mathbf{x} - \mathbf{U}_p + \mathbf{u}) + \hat{A} \cdot (-\dot{\mathbf{U}}_p + \dot{\mathbf{u}})) = 0\right\}$$

$$\hat{A} = \hat{A}(\alpha), \quad \alpha = \hat{A}^{-1} \cdot (\mathbf{x} - \mathbf{U}_p + \mathbf{u}) \tag{2.5}$$

Here, $\dot{\mathbf{u}}$ is the derivative with respect to the parameter t of the exact solution $\mathbf{u}$, a prime denotes a derivative with respect to the variable α, $\mathbf{U}_p$ and $\hat{A}$ are the translational displacement vector of the punch and the matrix of the rotation of the punch with respect to a certain fixed (laboratory) frame of reference and Ψ is a function which describes the surface of the punch and is used to formulate the unilateral constraints in definition (2.4) of the set of kinematically permissible velocities.

The resulting formulation is then used as follows.

1°. Replacing the derivatives with respect to the parameter by difference relations, we obtain a stepwise solution process which enables us to take account of the loading history.

2°. Introducing iterations with respect to the contact pressure $\boldsymbol{\sigma}_\mathrm{N}(\mathbf{u})$, in accordance with which the quantity $\boldsymbol{\sigma}_\mathrm{N}(\mathbf{u})$ is calculated using the displacements for the preceding iteration, it becomes possible to estimate the mutual effect of the friction forces and the normal contact pressure.[11,20] Note that the proposed iterations with respect to the contact pressure[11,20] enable one to transform each iteration to the problem of minimizing a functional with constraints.

3°. When equality (2.20 is used, each minimization problem is converted into to the problem of finding a saddle point which, in turn, is solved using the modified Udzawa method indicated above.

Different contact problems have been solved using this technique, and descriptions of these have been collected in the monograph Ref. 11.

3. Some generalizations, new and unsolved problems

Problems, which are attracting attention at the present time, can be roughly combined using criteria, that is, the names of the subsections of this section. Typical publications, which have already become classical, or even original contemporary publications are pointed out in each of them. The list of these publications could easily be continued, increasing them by an order of magnitude. However, the aim of the present paper is to demonstrate the fact that research on the contact problem remains timely, and continues to be carried out over a broad front in very different directions of both mathematics and mechanics.

3.1. Mathematical investigations

Contact problems are considered in mathematics as a basis for abstract generalizations, after which the theorems which have been proved are used to analyse some or other properties of the solutions. The researches in Refs 21–25 can be mentioned as examples of papers in this direction. An extension of the results referring to Signorini's problem is given in Ref. 26. Note that, as a rule, there are no solutions of specific problems here.

The problem of the possible non-uniqueness of the solution of a contact problem with friction has been studied[27] and an analogy with the corresponding theory of the bifurcation of solutions in the theory of plastic flow developed by Hill has been used.

3.2. The problem of rolling

The problem of rolling, which is very important in applications, continues to attract attention. The generally accepted approach to solving rolling problems is the use of a hypothesis concerning the linearity of the material and the independence of the friction forces from the normal pressure, which enables the relation between the friction forces and the relative slippage velocities to be found. After this, theorems are proved on the equivalence of the problem in a local formulation and the problem of minimizing a certain functional with constraints and convex optimization methods which have been developed are used to solve it. An account of the results obtained by this route can be found Refs 28,29. Papers in this area are continuing to appear for example, Ref. 30, in which two-dimensional rolling problems are solved using the boundary-element method.

3.3. Boundary conditions

Investigations into the problem of formulating boundary conditions in the contact area continue. While the problem can be considered as having been solved for small displacements and deformations, the problem turns out to be much less simple in the case of finite displacements and deformations. Heuristic algorithms of a different kind, which satisfy the no-penetration condition for finite deformation and the law of friction, are used in the numerical solution of problems. The most well-known of these algorithms involves interpolation of the displacements of selected nodes of one of the contacting bodies at the contact point, which is a node on the boundary of the second body. This algorithm is usually used in combination with a stepwise procedure with respect to the load.[11,31] A stepwise method with respect to the load with a calculation of the gap in the step, as is done in the geometrically linear theory, has been used[32] to solve the problem of the drawing of a wire from a cylindrical billet in a geometrically non-linear formulation.

The use of contact conditions (taking account of friction) of the "Asegment - segment" type[33] serves as an example of a more up-to-date technique in which the question is not about the conditions at a point (node) but along the whole edge of a finite element near the boundary. This technique is effective in the case of a large number of finite elements and, correspondingly, when supercomputers are used.

The theoretical investigation of contact conditions in the case of finite deformations[34,35] is based on the use of the geometry of non-Euclidean surfaces: the metrics of two different surfaces, that is, the metrics of the boundaries of the deformable bodies in their current state, are matched in the contact area. At present, there are no examples of the use of this approach to solve specific contact problems.

Papers, that deal with the generalization of the laws of interaction of surfaces in contact and, in particular, take account of adhesive forces, are part of this trend. The pioneering paper here was Ref. 36. A general variational theory has been developed.[37,38] We mention the later paper in this area.[11,39] The laws of viscoelastic-type slippage as applied to the problem of the modelling of earthquakes have been studied.[40]

The proof of a weak solution of a contact problem with friction has been proposed[41] when the coefficient of friction is an arbitrary function of the tangential displacements. The deficiencies in this formulation have been noted above although the apparatus itself, when supplemented with an algorithm to take account of the no-penetration conditions, can be generalized to laws of friction of the velocity type.

A theory of friction has been developed when the contact surface is anisotropic, which uses the method of the associated law in the theory of plastic flow.[42] New variational inequalities for contact problems with friction have been constructed and studied.[43,44]

3.4. Inelastic materials

There is a review of papers dealing with contact problems for linearly viscoelastic and ideally plastic media[11,5] and, also, for materials which obey the Henke-Ilyushin hardening law.[11] A proof of the weak solution of Signorini's contact problem with friction of the velocity type and for a Voigt material has been given.[45]

3.5. Numerical algorithms and their substantiation

A large number of papers are devoted to the testing of different methods of discretization and to the application and substantiation of methods for solving systems of discrete inequalities. Numerical algorithms, which have since become classical algorithms, are contained in a monograph.[8] The method of local variations[46] was one of the first numerical methods for solving problems involving the search for an extremum when there are constraints in the form of inequalities. Of the present-day papers, we single out the investigations, devoted to the use of adaptive nets,[47] which enable one to describe large stress gradients in the neighbourhood of the contact area more accurately.

A large amount of attention has also been given to the development of algorithms for solving problems of contact dynamics. Formally, these problems can be solved using Ostrogradskii's method but problems arise in its practical implementation, which are caused both by the variability of the contact, adhesion and slippage zones as well as by wave processes. The first problem was solved by Czekanski et al[48] using an iterative process (without proof of convergence) and the Chang and Hulbert scheme was used for the integration with respect to time. An implicit difference scheme was proposed for solving a similar problem and its convergence was proved.[49]

Analytical investigations of problems in contact dynamics have been carried out.[50] Signorini's problem, extended to the case of dynamic contact interaction with friction, has been studied.[51] An algorithm for the deparallelization of the calculations in combination with the method of decomposition of the zones has been proposed and implemented, [52] and model two-dimensional problems and the problem of designing springs have been solved. Conservative integration schemes have been proposed and one-dimensional problems and the problem of the collision of circular domains (discs) have been solved.[53]

3.6. Contact of electroelastic materials

The construction of micromachines such as micromotors and micromanipulators required the solution of problems concerning the contact of elements of the micromachines, the deformation of which is caused by the application of an external electric field. The theory of such materials and the solutions of a number of important practical problems have been gathered together in a monograph.[54]

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